Lecture Notes on
Analemma

Fahim Rajit Hossain
farahoshwadhin.13@gmail.com

These notes cover basic analemma, EoT, Kepler's equation, and sundials with a focus on Astronomy Olympiads. I have included discussion of astronomical methodology, but much of it is just my personal opinion.

> If a question on an astronomy exam starts with the phrase "We see analemma. . ." then the answer is probably "Equation of Time".

Contents

1 Introduction

You have surely heard the word *analemma* and seen its small curved shape in astronomical photographs. But until now, have you ever thought about what astronomical reality lies behind it? What reason does it have that the Sun traces such a strange and astonishing path over a year? To be honest, why does the Sun move in a curve at all? What does it relate to?

We want to discuss this phenomenon in a simple manner and answer these important questions of human curiosity. For this reason, we will also talk about the dependence of the analemma on the equation of time, and later we will examine this quantity and become familiar with its method of calculation.

But the main point is that the use of the analemma in astronomy is not very extensive; it currently does not have many practical questions or applications. Among astronomical books, the analemma is mentioned and analyzed only in a few. Of course, some books provide sufficient explanation so that one can reach the conclusion independently. Nevertheless, it does not receive detailed treatment in Olympiad-level books. Perhaps this is because of its complexity, or because it leads to many good ideas and beautiful questions.

2 Background

Analemma is a word that is said to mean "pedestal of a sundial."

The story is that if you record the position of your shadow cast by a gnomon at a specific time each day, and continue this work for a year, you will obtain a closed figure similar to the English number 8. This is the Sun's analemma.

You can perform this experiment using a simple camera and a sturdy tripod, or even with a smartphone. In this method, the device is fixed, and you photograph the Sun at the same hour each day. After completing one full year, if your timing and alignment were precise, you will obtain a figure similar to the one shown above. A few years ago, a photographer carried this out with remarkable precision and produced a very beautiful image of the Sun's analemma. An interesting point is that the analemma is created solely because of human definitions and conventions. Therefore, it can be considered a semi-artificial phenomenon. When we photograph the Sun at a specific fixed time, we are, in fact, imposing a condition on the shape of the analemma. The wristwatch we use is itself based on human conventions, and this causes the Sun to be observed each day at a particular point in the sky.

With this introduction, we begin the main discussion. First, we introduce a quantity called the equation of time, which serves as a bridge between astronomical quantities and human-defined clock time. The analemma phenomenon is directly related to the equation of time.

Before We Begin

It is assumed that the reader is familiar with the fundamental concepts of spherical astronomy and celestial mechanics. To become acquainted with these topics, it is recommended that the following sources be studied prior to proceeding:

- *Spherical Astronomy* by W. M. Smart — Chapters 1, 2, 4, and 5.
- *Orbital Mechanics* by Howard D. Curtis — Chapters 2, 3, and 4.

In most situations, reasoning in an Earth-centered frame is equivalent to reasoning in a Sun-centered frame. For convenience, in some parts of this text, we may treat the Earth as the center.

3   Equation of Time

Human beings require an accurate and repeatable scheduling system for their activities. For this reason, the design of a timekeeping system is of great importance.

In ancient times, people measured time by observing the Sun. In this approach, a "day" was defined as the interval between two successive meridian transits of the Sun (solar noon). However, modern humans encounter problems with such a system. (You can imagine what would happen if, at the moment of true solar noon every day, your watch showed exactly 12:00. What problems would arise?) The main source of these issues is that the Sun moves, and its motion is highly complex. The two primary causes of this complexity are:

1. The Earth's orbit around the Sun is elliptical. Therefore, Earth's orbital speed, and consequently the Sun's apparent speed in the sky, varies.
2. The rotational and orbital planes of the Earth are inclined relative to each other. In other words, the Sun moves along the ecliptic, whereas our clocks are based on the celestial equator.

To resolve these two problems, we imagine a fictitious object called the *mean Sun*. This hypothetical Sun has two fundamental properties: (1) it moves with constant angular speed (equal to the average apparent speed of the real Sun), and (2) it moves along the celestial equator. The first property is achieved by the *equation of the center*, and the second by the *reduction to the equator*.

In the equation of the center, we define a Sun (the dynamical mean Sun) that moves in a circular orbit around the Earth. Hence, its angular speed in the sky is constant. This imaginary Sun begins to move on the ecliptic when the real Sun is at perihelion.

When the dynamical mean Sun reaches the vernal equinox, the mean Sun, moving along the celestial equator with constant speed, begins its motion.

It is clear that the right ascension of the mean Sun and the real Sun will differ. The difference in their right ascensions is called the *Equation of Time*, denoted by E:

$$Eqt = \mathrm{RA_{MS}} - \mathrm{RA_{\odot}}.$$

If we know the right ascension of the real Sun, we can compute the right ascension of the mean Sun from the above relation. However, right ascension is not directly measurable, whereas the hour angle is a more accessible quantity. At any moment, given the Sun's azimuth and altitude, the hour angle can be determined, and with the equation of time, the hour angle of the mean Sun may be obtained. From the definition of local sidereal time (LST), we have:

$$\mathrm{LST} = \mathrm{RA} + \mathrm{HA} \implies Eqt = \mathrm{HA_{\odot}} - \mathrm{HA_{MS}}.$$

Civil time (the time shown on a wristwatch) is related through:

$$MT = \text{HA}_{\text{MS}} + 12^{\text{h}}.$$

Naturally, one must also apply the correction corresponding to the difference between the observer's longitude and the standard meridian.

Based on this convention, midnight (00:00) occurs when the mean Sun is at its lower culmination. This provides all the relations we need. Using the equation of time, we can predict the civil time at which an astronomical event will occur (an example of this calculation will be provided at the end of this lesson). For now, we set aside civil time and proceed to discuss the equation of time and its method of calculation.

3.1   Computing the Equation of Time

As you will see, for an exact computation of the Equation of Time, we cannot write a closed-form expression; therefore, we must rely on numerical methods.

We compute the Equation of Time for each day of the year in the following order:

We first calculate the mean dynamical longitude of the Sun along the ecliptic. Approximately, the Earth travels the full circle of the ecliptic in one year; therefore, it moves about $1°$ per day.

$$M = \frac{2\pi}{T_{\oplus}} \times \Delta t = \frac{2\pi}{T_{\oplus}} \times (D + \varpi), \qquad T_{\oplus} = 365.25$$

In this expression, D is the number of days elapsed since the first day of spring, and $\varpi \approx 75.5°$ is the angular distance between perihelion and the vernal equinox. In the equation above, M is the *mean anomaly*, one of the parameters of an elliptical orbit, and it is evaluated for the day in question.

We must determine the Sun's true (geometric) longitude on that day. Since the Earth's orbit is elliptical, the necessary computation must be carried out on an elliptical orbit with the Sun at one focus. Later we will write the full expression, but because the eccentricity of Earth's orbit is small ($e = 0.0167$), we can ignore part of the formula for now and return to the complete expression afterward.

From Kepler's equation for an elliptical orbit (with the Sun at the focus),

$$M = E - e \sin E,$$

we obtain the eccentric anomaly E.

Since solving this equation for E requires iterative numerical methods, we will apply numerical techniques such as the Newton–Raphson method or the bisection method. Through an example later in this text, the procedure for computing the Equation of Time will become clear.

Because we cannot write a closed-form expression for the Equation of Time, this explains why the relation above appears inside the parentheses. After computing the eccentric anomaly E, we obtain the Sun's true anomaly ν from:

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}}\,\tan\left(\frac{E}{2}\right).$$

The true geometric longitude of the Sun is related to Earth's true anomaly as shown in the figure:
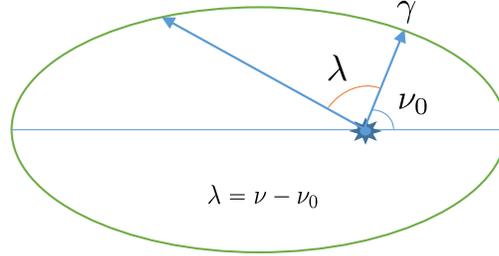
where $\nu_0 \approx 76°20'$ is the Sun's true anomaly at the vernal equinox.

Now that we have obtained the Sun's true ecliptic longitude for the day in question, we convert it into celestial coordinates using spherical relations. In the spherical triangle shown below, the following identities may be used:
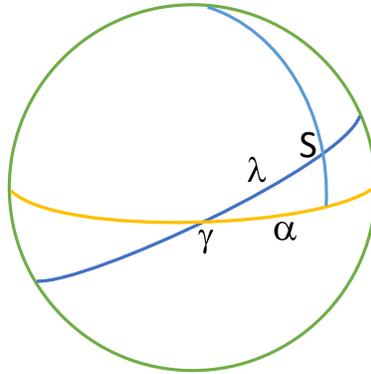
$$\cos\alpha\cos\varepsilon = \sin\alpha\cot\lambda - \sin\varepsilon\cot 90° \implies \tan\alpha = \cos\varepsilon\tan\lambda.$$

Thus, the right ascension of the Sun becomes:

$$\mathrm{RA}_\odot = \tan^{-1}\left(\cos\varepsilon\tan\lambda\right).$$

At this point we have the true right ascension of the Sun for the given day. Therefore, when computing the Equation of Time relative to the dynamical mean Sun located at the vernal equinox, we can proceed directly.

The final step is to compute the Sun's declination; however, since the Sun lies on the ecliptic and we have the standard spherical relations for converting from ecliptic to equatorial coordinates, this can be carried out straightforwardly.

When the real Sun is at perihelion, the mean Sun begins its motion from the point ν_0 on the celestial equator. The true anomaly of the real Sun at the vernal equinox, or the angular separation between perihelion and the vernal equinox, is shown in the diagram for clarity.
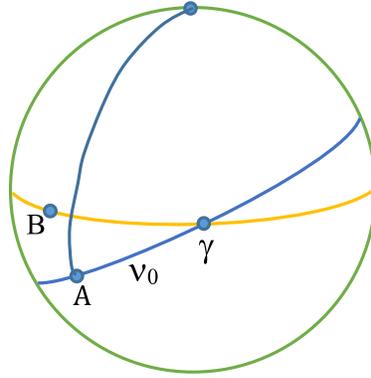
From the fact that the mean Sun moves at constant angular speed, the amount of time required for it to sweep a certain angular distance is proportional to that distance. Hence, the time corresponding to the Sun's true anomaly ν gives the arc on the equator traced by the mean Sun. Therefore:

$$\text{mean solar longitude} = M - \nu_0.$$

Now that we have the initial position of the mean Sun, we can easily obtain its right ascension for any given day by applying a small correction. For the day under consideration:

$$\alpha_0 = M - \nu_0 \implies \text{RA}_{\text{MS}} = M - \nu_0.$$

Thus, the Equation of Time becomes:

$$Eqt = \text{RA}_{\text{MS}} - \text{RA}_{\odot} = (M - \nu_0) - \tan^{-1}(\cos \varepsilon \, \tan(\nu - \nu_0)).$$

Example 3.1

The *equation of time* (Eqt) is historically a very important tool in astrometry, determining the difference between the *true solar time* and the *mean solar time*:

$$Eqt = H - H_m.$$

In this problem, we obtain a first approximation to the equation of time in terms of the orbital eccentricity.

  a. Let $t_0 = 0$ be the instant when the Sun crosses the vernal equinox. Find the equation of time Eqt as a function of time t and of the Sun's ecliptic longitude λ.
  b. In the low-eccentricity approximation (i.e., neglecting terms of order e^2), find the true anomaly of the planet as a function of time, assuming that the Earth takes Δt days to travel from perihelion to the March equinox.
  c. Assuming $\Delta t = 87\,\text{days}$, determine the equation of time Eqt as a function of time t only.

(a) For any location on Earth, the difference between the true solar time and the mean solar time is

$$
\begin{aligned}
Eqt &= H - H_m \\
&= (\text{LST} - RA) - (\text{LST} - RA_m) \\
&= \omega t - RA,
\end{aligned}
$$

where
$$\omega = \frac{360°}{T_\oplus}.$$

We now consider the spherical triangle formed by the north celestial pole P_N, the north ecliptic pole P_E, and the Sun P_O. Since the vernal equinox lies both on the celestial equator and on the ecliptic, we have on the celestial sphere

$$\angle P_O P_E P_N = 90° - \lambda,$$
$$\angle P_E P_N P_O = 90° + RA,$$

where λ is the ecliptic longitude of the Sun and RA is its right ascension.
Applying the four-elements formula of spherical trigonometry, we obtain

$$\cot(90°) \sin\varepsilon = \cot(90° + RA) \sin(90° - \lambda) + \cos\varepsilon \cos(90° - \lambda).$$

This expression simplifies to
$$\tan RA = \cos\varepsilon \tan\lambda.$$

Therefore, the equation of time can be written as

$$Eqt = \omega t - \tan^{-1}(\cos\varepsilon \tan\lambda).$$

(b) From Kepler's equation, the mean anomaly is related to the eccentric anomaly by

$$M = E - e\sin E.$$

Since $|\sin E| \leq 1$, it follows that $(E - M)\,e \approx 0$. Therefore,

$$M = E - e\sin\big(M + (E - M)\big)$$
$$= E - e\sin M\cos(E - M) - e\cos M\sin(E - M).$$

Taking $\cos(E - M) \simeq 1$ and $\sin(E - M)\,e \sim e^2 \approx 0$, we obtain

$$M \simeq E - e\sin M.$$

Since M is measured from perihelion, we may write

$$M = \omega(t + \Delta t),$$

and therefore
$$Eqt(t) = \omega(t + \Delta t) + e\sin\big(\omega(t + \Delta t)\big).$$

Relating this to the true anomaly ν, we use

$$\left(\frac{r}{a}\right) = 1 - e\cos E = \frac{1 - e^2}{1 + e\cos\nu} \implies 1 + e\cos\nu = \frac{1 - e^2}{1 - e\cos E}.$$

Thus,
$$\cos\nu = \frac{\cos E - e}{1 - e\cos E} \simeq \cos E - e(1 - \cos E) \simeq \cos\left(E + e\,\frac{1 - \cos E}{\sin E}\right).$$

This leads to

$$\nu(t) = E(t) + e\,\frac{1 - \cos E(t)}{\sin E(t)}.$$

Since $E - M \sim e$, we may approximate once more and write

$$\nu(t) = E(t) + e\,\frac{1 - \cos\bigl(\omega(t + \Delta t)\bigr)}{\sin\bigl(\omega(t + \Delta t)\bigr)}.$$

Therefore,

$$\boxed{\nu(t) = \omega(t + \Delta t) + e\sin\bigl(\omega(t + \Delta t)\bigr) + e\,\frac{1 - \cos\bigl(\omega(t + \Delta t)\bigr)}{\sin\bigl(\omega(t + \Delta t)\bigr)}.}$$

(c) Since we frequently work with angular quantities in radians, we convert

$$85°45' = 1.497.$$

Under these conditions, we can find an explicit formula for the ecliptic longitude λ, since the Sun always lies on the ecliptic:

$$\lambda(t) = \nu(t) - \nu(0).$$

Using the expression obtained previously for the true anomaly, we have

$$\lambda(t) = \omega t + e\sin(\omega t + 1.497) + e\,\frac{1 - \cos(\omega t + 1.497)}{\sin(\omega t + 1.497)}$$
$$- e\left[\sin(1.497) + \frac{1 - \cos(1.497)}{\sin(1.497)}\right].$$

Evaluating the constant term inside the brackets, this becomes

$$\lambda(t) = \omega t + e\left(\sin(\omega t + 1.497) + \frac{1 - \cos(\omega t + 1.497)}{\sin(\omega t + 1.497)} - 1.926\right).$$

Substituting this result into the equation of time, we obtain

$$Eqt(t) = \omega t - \tan^{-1}\left\{\cos\varepsilon\,\tan\left[\omega t + e\left(\sin(\omega t + 1.497) + \frac{1 - \cos(\omega t + 1.497)}{\sin(\omega t + 1.497)} - 1.926\right)\right]\right\}.$$

Example 3.2

Compute the Equation of Time for the first day of June. Let $\nu_0 = 76°20'$ and assume perihelion occurs 14 days after the winter solstice.

We first calculate the mean anomaly[a]:

$$M = \frac{2\pi}{T_e}(D + \varpi) = \frac{2\pi}{365.25}(62 + 75.5) = 2.3653° \approx 2.36°.$$

To compute the eccentric anomaly from Kepler's equation, we apply the Newton–Raphson method. We rewrite the problem as finding the root of the function

$$y = E - e \sin E - M.$$

We seek the value of E for which y is approximately zero. The method involves iterative refinement up to four decimal places. For the first iteration we set $E_1 = M$, compute y_1, and refine the estimate. Using the improved value, we compute E_2, and continue until convergence:

$$E_i: \quad y_i = E_i - e \sin E_i - M, \qquad (|y_i| < \varepsilon) \Rightarrow E = E_i,$$

$$y_i' = 1 - e \cos E_i, \qquad E_{i+1} = E_i - \frac{y_i}{y_i'}.$$

After several iterations, the solution for the date under consideration becomes:

$$E = 2.3769.$$

Next, we compute the Sun's true anomaly:

$$\nu = 2 \tan^{-1}\left(\sqrt{\frac{1 + 0.0167}{1 - 0.0167}} \times \tan\left(\frac{2.3769}{2}\right)\right) = 2.3884 \approx 2.39°.$$

We now obtain the Sun's true ecliptic longitude:

$$\lambda_\odot = \nu - \nu_0 = 2.3884 - 1.3322 = 1.0562 \, \text{rad} = 60.516° \approx 60.52°.$$

The Sun's true right ascension is:

$$\text{RA}_\odot = \tan^{-1}(\cos 23.45° \tan 60.516°) = 58.35°.$$

The mean Sun's right ascension becomes:

$$\text{RA}_{\text{MS}} = M - \nu_0 = 2.3653 - 1.3322 = 1.0331 \, \text{rad} = 59.19°.$$

Finally, the Equation of Time is:

$$Eqt = \text{RA}_{\text{MS}} - \text{RA}_\odot = 59.192 - 58.354 = 0.84°,$$

which corresponds to approximately 3.2 minutes.

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[a]The same method is used as before; we keep these approximate values and proceed with the calculation.

3.2 Equation of Time Curve

The *equation of time* is defined as the difference between the true solar time and the mean solar time,

$$Eqt = H - H_m = RA_m - RA,$$

where RA is the right ascension of the true Sun and RA_m is the right ascension of the mean Sun. Since the mean Sun moves uniformly along the celestial equator, its right ascension increases linearly with time,

$$RA_m = \omega t, \qquad \omega = \frac{2\pi}{T_\oplus}.$$

Hence,

$$Eqt(t) = \omega t - RA(t).$$

Throughout this section, we define

$$\alpha(t) \equiv Eqt(t),$$

and express α in units of time (minutes).

Decomposition of the Equation of Time

To first order in the small parameters ε (obliquity of the ecliptic) and e (orbital eccentricity), the equation of time separates into two additive contributions,

$$\alpha(t) = \alpha_{\text{tilt}}(t) + \alpha_{\text{ecc}}(t).$$

This decomposition reflects two independent effects:

- the projection of uniform motion along the ecliptic onto the equator (axial tilt),
- the non-uniform orbital motion of the Earth (eccentricity).

Solar Declination

Let n denote the day of the year, with $n = 1$ corresponding to January 1. The solar declination varies periodically with period one year. A convenient approximation is

$$\boxed{\delta(n) = -23.44° \cos\left(2\pi\frac{n+10}{365}\right),}$$

which places the minimum declination at the winter solstice and the maximum at the summer solstice.

Axial-Tilt Contribution

For a body moving uniformly along the ecliptic, the transformation between ecliptic longitude λ and right ascension yields, for small obliquity,

$$RA \simeq \lambda - \frac{\varepsilon^2}{4}\sin 2\lambda.$$

The second term represents a deviation from uniform motion and produces a contribution to the equation of time with twice the annual frequency. Thus,

$$\alpha_{\text{tilt}}(n) = -A_{\text{tilt}} \sin\left(4\pi \frac{n+10}{365}\right),$$

where the angular amplitude is

$$A_{\text{tilt}} = \tan^2\left(\frac{\varepsilon}{2}\right).$$

Converting from radians to minutes using

$$1 \text{ rad} = \frac{1440}{2\pi} \text{ min},$$

the amplitude for Earth becomes

$$A_{\text{tilt}} \approx 9.8 \text{ min.}$$

Eccentricity Contribution

Kepler's second law implies that the Sun's angular speed along the ecliptic is not constant. To first order in eccentricity, the ecliptic longitude may be written as

$$\lambda(t) = \omega t + 2e\sin(\omega t + \lambda_0),$$

where λ_0 fixes the position of perihelion.
This produces a contribution to the equation of time with period one year,

$$\alpha_{\text{ecc}}(n) = -A_{\text{ecc}} \sin\left(2\pi \frac{n-n_p}{365}\right),$$

where $n_p \approx 4$ corresponds to perihelion (January 4) and

$$A_{\text{ecc}} = 2e.$$

In time units,

$$A_{\text{ecc}} \approx 7.7 \text{ min.}$$

Total Equation of Time

The total equation of time is obtained by linear superposition,

$$\alpha(n) = \alpha_{\text{tilt}}(n) + \alpha_{\text{ecc}}(n).$$

Each contribution is plotted separately as a function of day number n, and their sum yields the characteristic asymmetric waveform of the equation of time. The horizontal axis is placed at $\alpha = 0$ to emphasize the dates at which true solar noon coincides with mean solar noon.

Mathematically, the asymmetry of the equation of time arises from the superposition of a half-year sinusoid (axial tilt) and a one-year sinusoid (eccentricity) with different amplitudes and phase offsets. This superposition underlies the shape of the solar analemma.
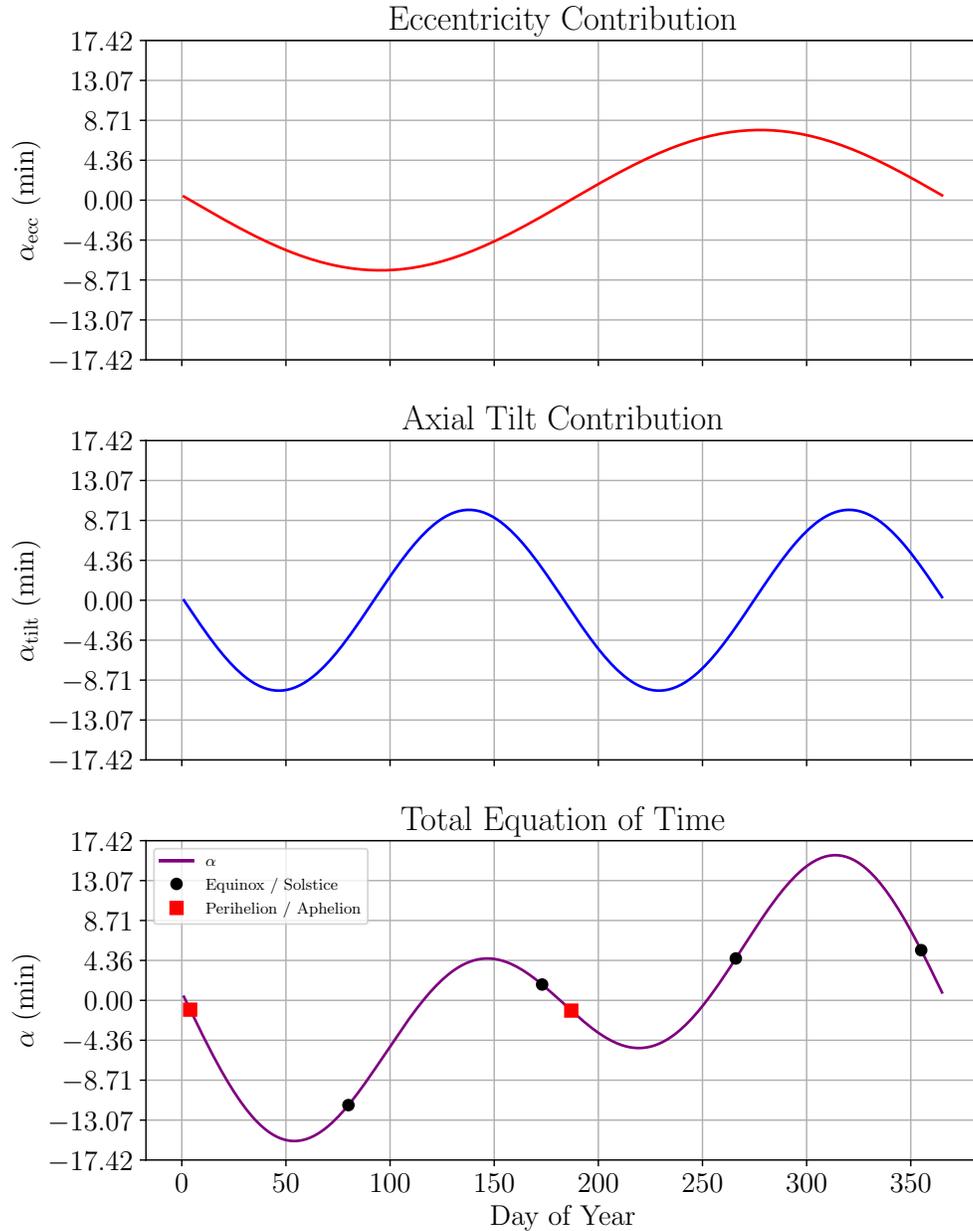
By examining the curve, we observe that it possesses no symmetry. Later, when we analyze the analemma itself, we will see that it also exhibits no intrinsic symmetry.

On the first day of Farvardin (approximately March 21), the Equation of Time is about -7 minutes (this is obtained by subtracting the true right ascension of the Sun at the vernal equinox from that of the mean Sun).

Four dates throughout the year yield an Equation of Time equal to zero. Notice that these days **do not** correspond to the solstices or the equinoxes.

It is evident that the Equation of Time varies between two extrema during the year: approximately

+16.4 minutes and −14.3 minutes. At the end of this lesson we will attempt to determine these critical points (local maxima and minima).

Below are several important points on the curve along with their approximate dates:

Date	Equation of Time (min)
Vernal equinox	−7.1
21 April	−1.2
30 April	−2.4
14 June	−2.7
1 August	−6.7
2 September	+0.1
3 November	+16.4
22 February	−14.3
3 January	+1.4

(The values above are only approximate; an uncertainty of roughly ±3 days should be assumed.) We now return to the analemma and analyze it using the Equation–of–Time curve.

Example 3.3 (*Shojiro-42*)

Bruno lives on a planet that orbits the star *Shojiro-42*. Being very curious, he decided to plot the *analemma* of his star, obtaining the figure shown above.
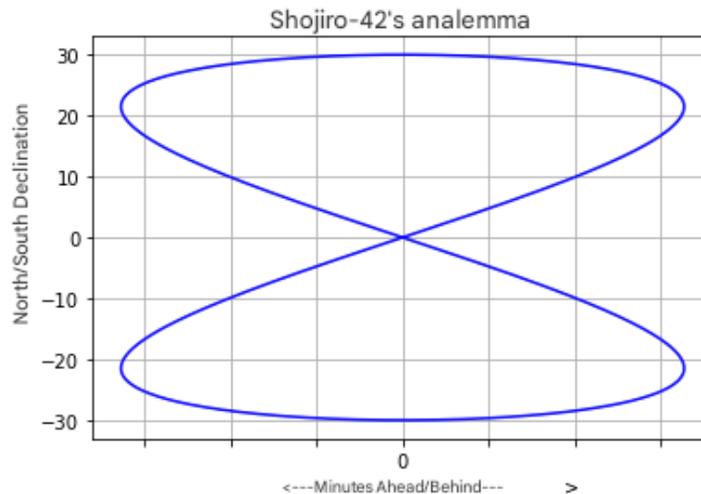
However, Bruno forgot to adopt a scale for the horizontal axis of his graph, which represents the *equation of time*. Help him with this task and determine the *magnitude of the largest possible value of the equation of time*, in minutes, on Bruno's planet.

Assume that one stellar day on Bruno's planet has a duration of 24 hours, and note that the eccentricity of its orbit is zero.

4   The Analemma

Having completed our discussion of the Equation of Time, we can now turn our attention to the analemma itself. As noted earlier, the analemma may be photographed by capturing the Sun at the same clock time every day over the course of a year.

Suppose the fixed observing time is 12:00 (clock noon). In that case, the mean Sun has an hour angle of zero. Recall the definition of civil time presented previously. We found that the Equation of Time can be expressed in terms of the hour angles of the Sun:

$$Eqt = HA_\odot - HA_{\mathrm{MS}}.$$

Setting $HA_{\mathrm{MS}} = 0$ implies that the Sun's true hour angle is equal to the Equation of Time. Thus, if we photograph the Sun every day at this fixed time, its displacement from the true meridian is governed by the Equation of Time.

We now understand why the Sun's horizontal displacement on the analemma changes: this is due to variations in the component of its motion associated with the obliquity of the ecliptic. The Sun's vertical displacement varies daily because of changes in its declination. The combination of these two varying components produces the full analemma.

Parametric Equation of Analemma [USAAO-24 National]

We parametrize time by the *mean anomaly M*, defined to increase uniformly from 0 to 2π over one orbital period, with $M = 0$ at perihelion. Since perihelion does not coincide with the vernal equinox, we define ν_0 to be the value of the anomaly at the vernal equinox, and introduce the *mean longitude*

$$L = M - \nu_0,$$

which corresponds to the longitude of the hypothetical mean Sun. Assuming the orbital eccentricity satisfies $e \ll 1$, we derive an expression for the Sun's *true ecliptic longitude* λ as a function of L and ν_0, without invoking Kepler's equation.

By definition, the mean anomaly is proportional to the area swept out by the radius vector from perihelion, so that

$$\frac{M}{2\pi} = \frac{\text{area swept since perihelion}}{\text{total orbital area}}.$$

By Kepler's second law, equal areas are swept in equal times. For a slightly eccentric orbit with $e \ll 1$, the radial distance of the Earth from the Sun as a function of the true anomaly ν is

$$r(\nu) = a(1 - e \cos \nu),$$

to first order in e, where a is the semi-major axis. The infinitesimal swept area is then

$$dA = \frac{1}{2} r^2 \, d\nu \simeq \frac{a^2}{2} \left(1 - 2e \cos \nu\right) d\nu.$$

Integrating from perihelion ($\nu = 0$) to a general true anomaly ν, the swept area is

$$A(\nu) = \frac{a^2}{2} \left(\nu - 2e \sin \nu\right).$$

Since the total orbital area is $A_{\text{tot}} = \pi a^2$, the definition of the mean anomaly gives

$$M = \nu - 2e \sin \nu,$$

valid to first order in e. Inverting this relation perturbatively yields

$$\nu = M + 2e \sin M.$$

The Sun's true ecliptic longitude λ is measured from the vernal equinox rather than from perihelion. Using $M = L + \nu_0$ and subtracting the offset ν_0, we obtain

$$\lambda = \nu - \nu_0 = (L + \nu_0 + 2e \sin(L + \nu_0)) - \nu_0.$$

Therefore, the Sun's true ecliptic longitude is

$$\boxed{\lambda(L, \nu_0) = L + 2e \sin(L + \nu_0),}$$

to first order in the orbital eccentricity. This result shows explicitly how the non-uniform orbital motion of the Earth introduces a periodic correction to the uniform motion of the mean Sun, an effect that contributes to the horizontal distortion of the analemma.

Derivation of the Analemma: Declination and Equation of Time

We describe the analemma as a parametric curve in terms of the mean longitude L, using the declination $\delta(L)$ and the equation of time

$$\Delta RA(L) = RA(L) - L,$$

where RA is the right ascension of the true Sun. Throughout this derivation we assume

$$e \ll \varepsilon \ll 1,$$

and retain only leading-order terms in these small parameters.
From part (a), the Sun's true ecliptic longitude is

$$\lambda(L) = L + 2e \sin(L + \nu_0).$$

Declination

From spherical trigonometry, the declination of a body lying on the ecliptic satisfies

$$\sin \delta = \sin \varepsilon \sin \lambda.$$

Using the small-angle approximations $\sin \varepsilon \simeq \varepsilon$ and $\sin \delta \simeq \delta$, we obtain

$$\delta \simeq \varepsilon \sin \lambda.$$

Substituting the expression for $\lambda(L)$,

$$\delta(L) \simeq \varepsilon \sin\big(L + 2e \sin(L + \nu_0)\big).$$

Since the correction term is already first order in e, we expand the sine to linear order,

$$\sin(A + \epsilon) \simeq \sin A + \epsilon \cos A,$$

which yields

$$\delta(L) \simeq \varepsilon \sin L + 2e\varepsilon \sin(L + \nu_0) \cos L.$$

The second term is of order $e\varepsilon$, which is second order overall and therefore neglected. Hence, to leading order,

$$\boxed{\delta(L) \simeq \varepsilon \sin L.}$$

Equation of Time

The right ascension RA is related to the ecliptic longitude by

$$\tan RA = \cos \varepsilon \tan \lambda.$$

Using $\cos \varepsilon \simeq 1 - \varepsilon^2/2$, we write

$$\tan RA \simeq \left(1 - \frac{\varepsilon^2}{2}\right) \tan \lambda.$$

For small differences between RA and λ, this implies

$$RA \simeq \lambda - \frac{\varepsilon^2}{2} \sin \lambda \cos \lambda = \lambda - \frac{\varepsilon^2}{4} \sin 2\lambda.$$

Substituting $\lambda(L) = L + 2e \sin(L + \nu_0)$ and replacing λ by L inside the sine (which introduces only higher-order corrections), we obtain

$$RA(L) \simeq L + 2e \sin(L + \nu_0) - \frac{\varepsilon^2}{4} \sin 2L.$$
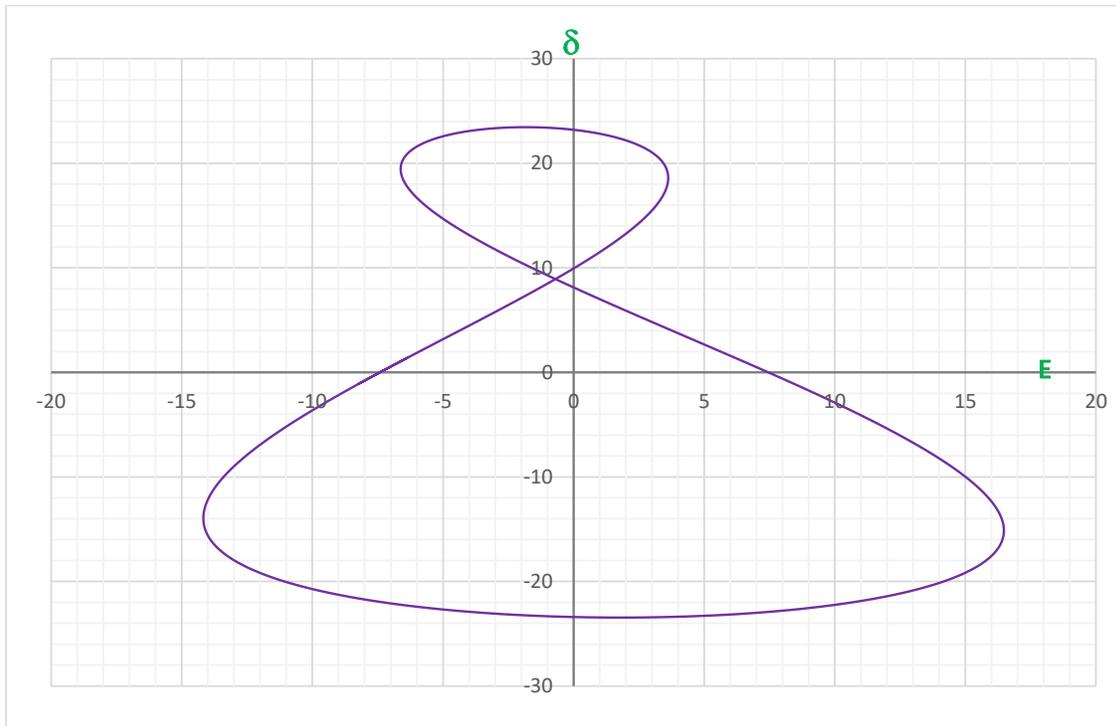
The equation of time is therefore

$$\boxed{\Delta RA(L) = 2e \sin(L + \nu_0) - \frac{\varepsilon^2}{4} \sin 2L.}$$

Together, the parametric equations

$$\delta(L) = \varepsilon \sin L, \qquad \Delta RA(L) = 2e \sin(L + \nu_0) - \frac{\varepsilon^2}{4} \sin 2L$$

describe the analytic form of the solar analemma to leading order in eccentricity and obliquity. We now proceed to plot a diagram and analyze how the Sun's declination and its horizontal motion together generate the analemma.

The curve shows a strong resemblance to the analemma; in fact, it is nearly identical. We can therefore proceed to analyze the analemma directly using this curve.

As mentioned earlier, the analemma possesses no symmetry. With the aid of the Equation–of–Time curve, we can now understand this more clearly. The table above indicates that the Equation of Time becomes zero on two dates. By examining the plot, we see that these points correspond to locations where the Sun crosses the meridian closest to true noon—that is, where the Sun's true hour angle is zero. These points arise directly from the Equation–of–Time curve.

Date (Gregorian)	Equation of Time (min)
March 23	-7
August 29	-7

A further observation from the plot is that the curve is stretched in two separate directions: one along the horizontal axis and another along the vertical axis. This stretching also contributes to the asymmetry of the analemma.

We previously noted that the zero points of the Equation of Time have no direct relation to the equinoxes or the solstices. The plot again confirms this: the shape of the curve depends only on the Sun's motion.

Throughout the year, the Sun follows the sequence illustrated by the curve: beginning near the horizontal axis with a negative Equation of Time, the loop ascends, crosses the axis, reaches a maximum, slows, continues toward the summer solstice, then descends, crosses zero again, reaches a minimum, and finally returns to the horizontal axis.

Thus, by examining the curve and identifying the approximate locations of the equinoxes and solstices, we can interpret the Sun's annual motion and relate it to the structure of the analemma.

5   Further Details About the Analemma

To understand the analemma more deeply, we now analyze its individual components. In general, we know that the Equation of Time determines the horizontal displacement of the analemma. Using numerical tools such as MATLAB or Excel, we can change the relevant parameters and observe how they affect the resulting shape.

In what follows, we will extract additional detailed information about the analemma.

Departure from the Center

The departure from Earth's real orbit corresponds to $e = 0.0167$. We observe that by varying e and keeping the other parameters fixed, the analemma takes on different shapes.
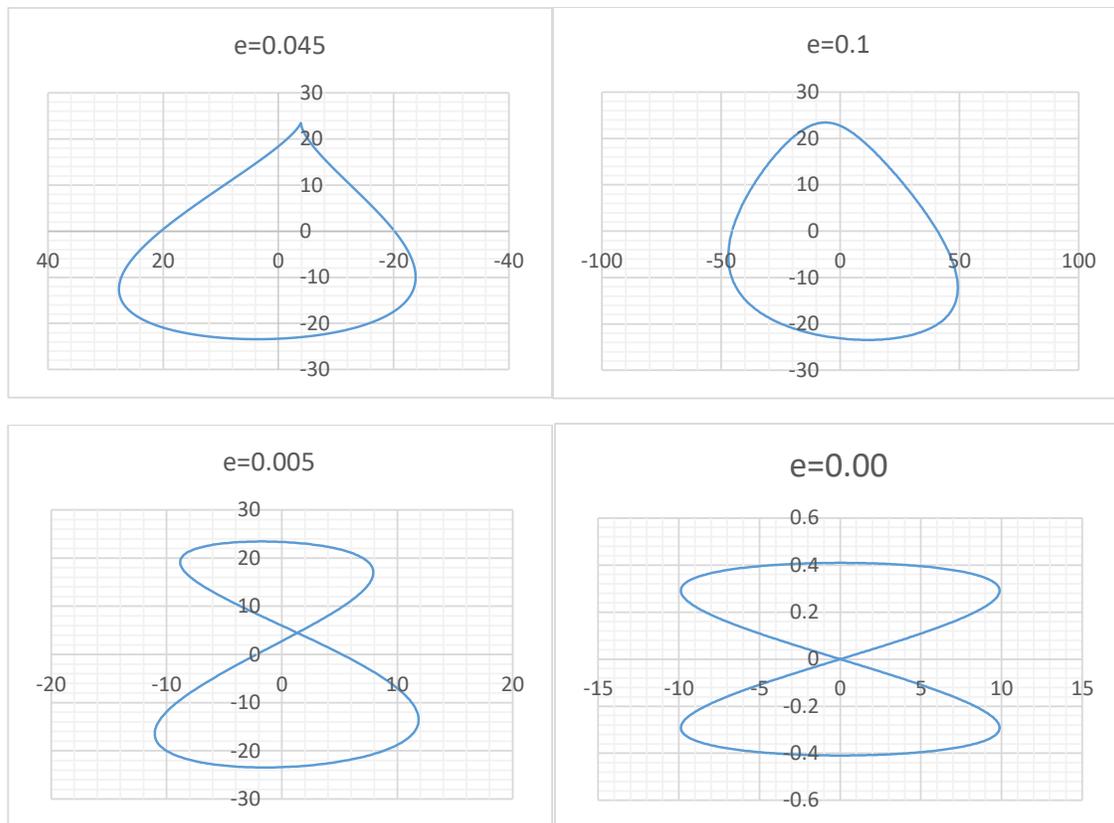

The Earth's analemma consists of two lobes. We observe that as the departure from the center increases, the upper lobe gradually disappears. In a special case, the analemma becomes teardrop-shaped. This special case depends on parameters other than eccentricity, including the longitude of perihelion. However, in all cases there exists a limit such that the analemma remains teardrop-shaped. If the departure from the center is increased beyond this limit, the teardrop shape opens outward and approaches a circular form.

The two lower plots show that as the departure from the center decreases, the analemma becomes more symmetric. In the case $e = 0$ (a circular orbit), the analemma is completely symmetric. As expected, we will examine this case further in the continuation of the calculations.

Ecliptic Longitude

By *ecliptic longitude* we mean the angular separation of the Earth from the vernal equinox. We know that the Earth reaches perihelion approximately 14 days after the winter solstice. Therefore, after the vernal equinox, it takes the Earth some time to reach perihelion. This delay causes the maximum of the equation of time to occur later than the vernal equinox. In a special case where we ignore the effect of each parameter separately, we can examine how the analemma changes shape as each parameter varies.
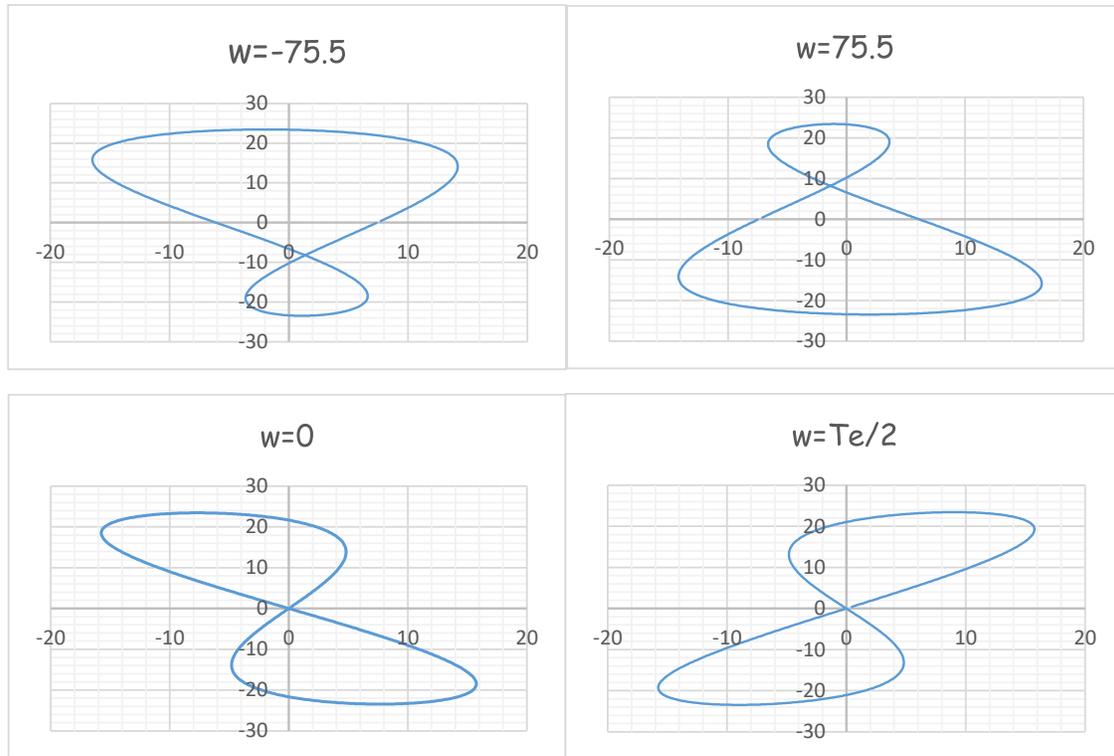
From the shapes shown in the figures, it is clear that the solutions are completely determined and there is no need for further discussion. In the special case of orbital eccentricity, the factor that changes the size and the left–right orientation of the analemma is the ecliptic longitude of perihelion.

Orbital Eccentricity

It can be stated that the most important factor in forming the shape of the analemma is the eccentricity of the orbit relative to the vernal equinox. The eccentricity of the Earth's orbit has a relatively small effect and is often neglected in simple calculations. Nevertheless, the eccentricity of the orbit cannot be completely ignored.

If this angle is equal to zero, it is clear that no analemma will exist; it will reduce to a straight line along the vertical axis.

On the other hand, if the axial tilt is equal to 90°, that is, the orbital plane is perpendicular to the equator, then again we will not have anything that can be called an analemma.

This is because, in one extreme, the difference between the mean Sun and the true Sun increases, and the analemma becomes as large as the entire sky. We have such conditions on the planet Uranus.

However, an important point regarding the remaining axial tilt can be seen in the diagram below:
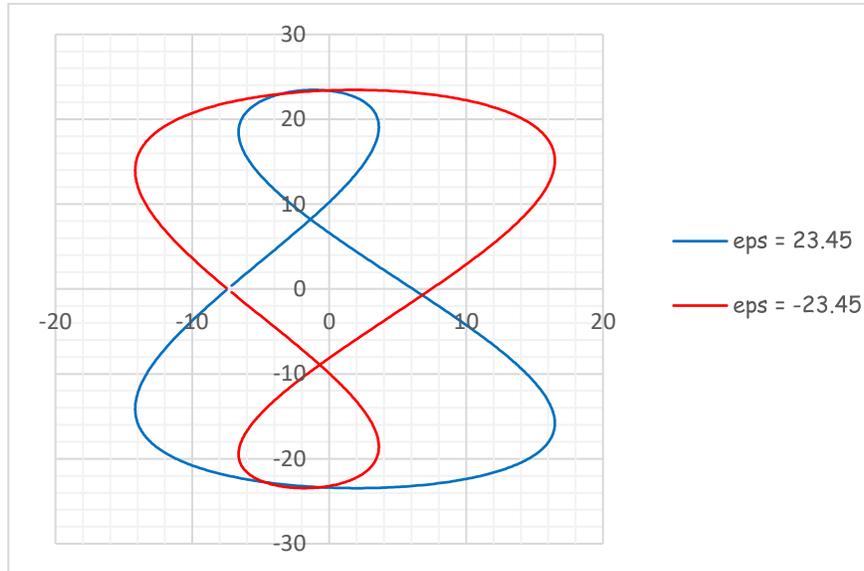
That is, in this problem, the case of zero axial tilt is symmetric with respect to the orbital plane.

Sidereal and Synodic Periods

An important point that must be mentioned here also concerns the main root of the discussion. The definition of a full day is the same classical definition. That is, one day is defined as the time interval between two successive transits of the true Sun across the local meridian, and it is equal to 24 hours. In this definition, the "day" is the solar day, not the sidereal day.

For the other planets, the solar day is defined in a similar manner. The method of calculating it is also analogous to the calculation of the synodic period of a planet relative to another planet. In this context, the method based on relative angular velocity is useful.

For example, a full rotation of Mars about its own axis corresponds to a Martian sidereal day equal to 1.0259 Earth days, that is, 24 hours and 37.5 minutes. By having the synodic (solar) period of Mars equal to 88,642 Earth seconds, a solar day on Mars lasts 24 hours and 39.5 minutes.

Using this information, we can imagine being on Mars and, over the duration of one Martian year, recording the analemma.

If we wish to estimate what the analemma of Mars might look like, we can make some educated guesses based on data obtained using astronomy software:

- Orbital eccentricity: 0.093
- Axial tilt relative to the orbital plane: 25.19°
- Time interval between perihelion and the vernal equinox: approximately 208 Earth days

First, note that this information—apart from the ecliptic longitude—is approximately similar to that of Earth. Therefore, the analemma of Mars can be larger. The larger orbital eccentricity may produce a teardrop-shaped analemma, meaning that the upper part of the analemma is suppressed. The axial tilt does not affect the overall size of the analemma but only changes its orientation. In general, we expect a shape similar to the one shown in the figure below.
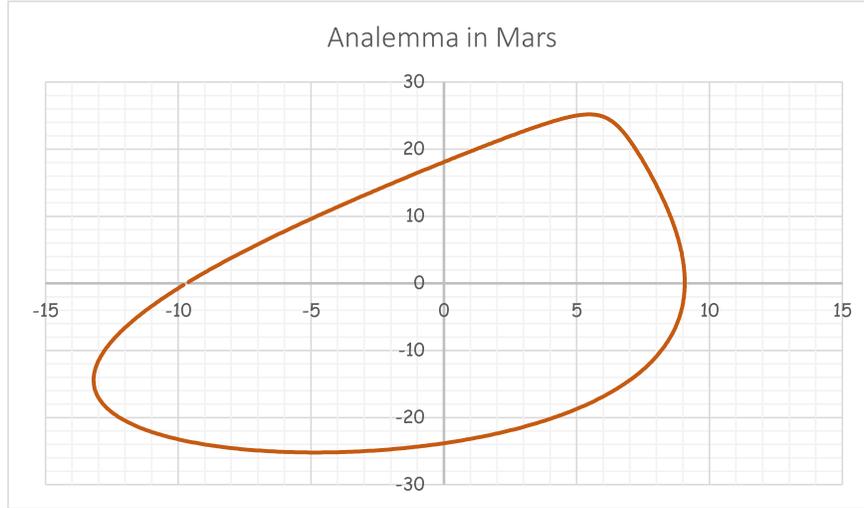
So far, we have examined the analemma in the most general case and have taken into account all the contributing factors. However, we can also make use of approximate methods. Although the use of approximations does not give the exact shape of the analemma, it provides a very good approximation. In what follows, we become familiar with simple approximations, and then we will examine a few remaining points related to the analemma.

Approximation of the Eccentricity

Among the factors affecting Earth's analemma, orbital eccentricity has a small effect and can be neglected. That is, we may set $e = 0$ and solve the problem. As a result, the eccentricity contribution is removed from the computation of the equation of time, and only the axial-tilt contribution remains. Using this simplification, we can analyze the problem using simple relations.

It is clear that when the eccentricity is zero, the concept of the longitude of perihelion no longer has any meaning. That is,

$$M = \frac{2\pi}{T_\oplus} \times \Delta t = \frac{2\pi}{T_\oplus} \times D = \Omega D,$$
$$RA_{\mathrm{MS}} = M,$$
$$\lambda_\odot = \nu = M.$$

Under the above conditions, the equation of time reaches zero at its extrema, so the only remaining task is to compute the right ascension of the true Sun. Thus, we have:

$$\tan RA_\odot = \cos \varepsilon \tan \lambda_\odot = \cos \varepsilon \tan M.$$

The equation of time is therefore

$$Eqt = RA_{\mathrm{MS}} - RA_\odot = M - \tan^{-1}(\cos \varepsilon \tan M).$$

We can rewrite the above expression as

$$\tan(M - Eqt) = \cos\varepsilon \tan M.$$

If all angles are written in radians, the equation of time becomes very small, and we may approximate it using a Taylor expansion. One may also solve the equation exactly without approximation, but the goal here is to obtain a simple analytic expression.

Keeping the equation of time to first order, we obtain

$$\tan M - Eqt(1 + \tan^2 M) + \cdots = \left(1 - \frac{\varepsilon^2}{2} + \cdots\right)\tan M.$$

This yields

$$\frac{Eqt}{\cos^2 M} = \frac{\varepsilon^2}{2}\tan M.$$

Hence,

$$Eqt = \frac{\varepsilon^2}{2}\sin M \cos M = \frac{\varepsilon^2}{4}\sin(2M).$$

The above equation is a periodic function of time and represents the axial-tilt contribution to the equation of time.

For the solar declination, we have

$$\sin\delta = \sin\varepsilon \sin\lambda = \sin\varepsilon \sin M.$$
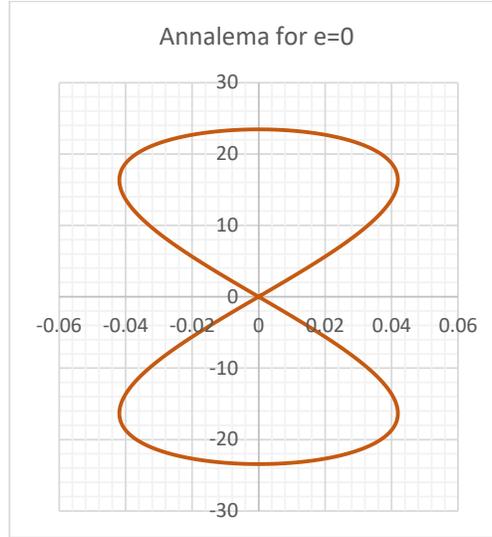
We now substitute for M:

$$Eqt = \frac{\varepsilon^2}{4}2\sin M \cos M = \frac{\varepsilon^2}{4}2\sin M \sqrt{1 - \sin^2 M}.$$

Writing the result in terms of the declination,

$$Eqt = \frac{\varepsilon^2}{4}\frac{2\sin\delta}{\sin\varepsilon}\sqrt{\frac{\sin^2\varepsilon - \sin^2\delta}{\sin^2\varepsilon}} = \frac{\varepsilon^2}{2}\frac{\sin\delta}{\sin^2\varepsilon}\sqrt{\sin^2\varepsilon - \sin^2\delta}.$$

The above expression may be further simplified using small-angle approximations, but it is already sufficient for plotting the curve.

Thus, plotting the solar declination versus the equation of time yields the curve shown below.

As we saw before, the graph is perfectly symmetrical.

Central Angle of the Analemma for the Case $e = 0$

We wish to compute the central angle of the analemma, that is, the angle between the two tangents to the analemma at the node, in the special case where the orbital eccentricity is zero. In this situation, the Sun moves with uniform angular speed along a circular orbit. Because the displacement of the Sun on the celestial sphere is very small, we may apply small-angle approximations to obtain the central angle. The shape of the analemma for the real Sun and for the mean Sun is shown. We know that the difference between the two Suns is exactly the equation of time.
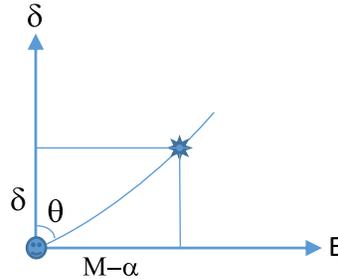
From the figure, the angle θ is obtained as:

$$\tan\theta = \frac{M - \alpha}{\delta}.$$

Using small-angle approximations, we have:

$$\delta \approx \sin\delta = \sin\varepsilon \sin M \approx M \sin\varepsilon,$$
$$\alpha \approx \tan\alpha = \cos\varepsilon \tan M \approx M \cos\varepsilon.$$

Using these relations, we find:

$$\tan\theta \approx \frac{M - M\cos\varepsilon}{M\sin\varepsilon} = \frac{1 - \cos\varepsilon}{\sin\varepsilon} = \frac{2\sin^2(\varepsilon/2)}{2\sin(\varepsilon/2)\cos(\varepsilon/2)} = \tan\frac{\varepsilon}{2}.$$

Therefore,

$$\theta = \frac{\varepsilon}{2}.$$

Thus, when $e = 0$, the analemma has a central angle equal to half the obliquity of the ecliptic.

It should be noted that the node angle of the analemma is only analytically tractable in the case where the orbital eccentricity is zero. In the general case, this angle must be computed numerically, which will be done later.

Critical Points of the Equation of Time

We have seen that the equation of time has both maxima and minima. In this section, we first examine each effect contributing to the equation of time separately, and then combine the discussions and present the calculations in the form of an example.

Example 5.1

> a. Prove that if we consider only axial tilt (i.e. $e = 0$), the extrema of the equation of time occur when the Sun's ecliptic longitude satisfies
>
> $$\lambda_\odot = \tan^{-1}\sqrt{\frac{1}{\cos\varepsilon}}.$$
>
> b. Prove that if we consider only orbital eccentricity (i.e. $\varepsilon = 0$), the extrema of the equation of time occur when the Sun's true anomaly is given by
>
> $$\nu_\odot = \cos^{-1}\left(\frac{(1-e^2)^{3/4} - 1}{e}\right).$$

(a) We write the equation of time and differentiate it:

$$Eqt = RA_{\mathrm{MS}} - RA_\odot = M - \alpha.$$

At the extrema of the equation of time,

$$\frac{dEqt}{dM} = 0 \implies \frac{d(M-\alpha)}{dM} = 0 \implies \frac{d\alpha}{dM} = 1.$$

Using

$$\tan\alpha = \cos\varepsilon\tan M,$$

we differentiate to obtain

$$(1 + \tan^2\alpha)\frac{d\alpha}{dM} = \cos\varepsilon\,(1 + \tan^2 M).$$

Thus,

$$\frac{d\alpha}{dM} = \frac{\cos\varepsilon\,(1 + \tan^2 M)}{1 + \cos^2\varepsilon\tan^2 M}.$$

By equating this result to unity and solving, we obtain

$$\tan^2 M = \frac{1}{\cos \varepsilon}.$$

By substituting the value of the axial tilt of Earth, the solar longitudes $47.7°$, $123.4°$, $237.7°$, and $312.5°$ are obtained, which correspond respectively to the dates April 17, June 10, August 15, and January 11. These dates are very close to the critical points of the equation-of-time curve.

(b) We start again from the equation of time:

$$Eqt = RA_{\mathrm{MS}} - RA_{\odot} = (M - \nu_0) - (\nu - \nu_0).$$

At the extrema of the equation of time we have

$$\frac{dEqt}{dM} = 0 \implies \frac{d\nu}{dM} = 1.$$

From Kepler's equation we differentiate:

$$M = E - e \sin E \implies \frac{dM}{dE} = 1 - e \cos E.$$

From the relation between the eccentric anomaly and the true anomaly we also differentiate:

$$\tan \frac{E}{2} = \beta \tan \frac{\nu}{2}, \qquad \text{where} \qquad \beta = \sqrt{\frac{1-e}{1+e}} = \frac{(1-e^2)^{1/2}}{1+e}.$$

Differentiating, we obtain

$$\left(1 + \tan^2 \frac{E}{2}\right) \frac{dE}{d\nu} = \beta \left(1 + \tan^2 \frac{\nu}{2}\right) \implies \frac{dE}{d\nu} = \frac{\beta \left(1 + \tan^2 \frac{\nu}{2}\right)}{1 + \beta^2 \tan^2 \frac{\nu}{2}}.$$

We continue:

$$\frac{dM}{d\nu} = \frac{dM}{dE}\frac{dE}{d\nu} = (1 - e \cos E)\frac{\beta \left(1 + \tan^2 \frac{\nu}{2}\right)}{1 + \beta^2 \tan^2 \frac{\nu}{2}}.$$

From the previous relation we can express $\cos E$ and substitute it into the equation above:

$$\frac{1 - \cos E}{1 + \cos E} = \beta^2 \tan^2 \frac{\nu}{2} \implies \cos E = \frac{1 - \beta^2 \tan^2 \frac{\nu}{2}}{1 + \beta^2 \tan^2 \frac{\nu}{2}}.$$

$$\frac{dM}{d\nu} = \left(1 - e\frac{1 - \beta^2 \tan^2 \frac{\nu}{2}}{1 + \beta^2 \tan^2 \frac{\nu}{2}}\right) \left(\frac{\beta \left(1 + \tan^2 \frac{\nu}{2}\right)}{1 + \beta^2 \tan^2 \frac{\nu}{2}}\right) = 1$$

Since

$$\tan^2 \frac{\nu}{2} = \frac{1 - \cos \nu}{1 + \cos \nu},$$

we simplify the relations above in terms of $\cos \nu$:

$$\frac{1 - \beta^2 \tan^2 \dfrac{\nu}{2}}{1 + \beta^2 \tan^2 \dfrac{\nu}{2}} = \frac{(1 - \beta^2) + (1 + \beta^2)\cos\nu}{(1 + \beta^2) + (1 - \beta^2)\cos\nu} = \frac{e + \cos\nu}{1 + e\cos\nu}$$

$$\frac{1 + \tan^2 \dfrac{\nu}{2}}{1 + \beta^2 \tan^2 \dfrac{\nu}{2}} = \frac{1 + e}{1 + e\cos\nu}$$
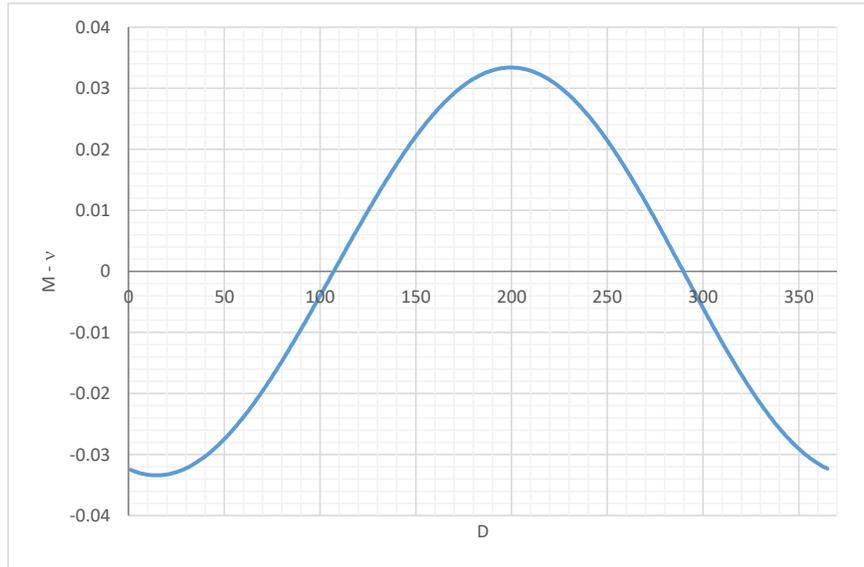
Substituting into the relation above:

$$\left(1 - e\frac{e + \cos\nu}{1 + e\cos\nu}\right)\left(\frac{1 + e}{1 + e\cos\nu}\right)\beta = 1$$

$$(1 + e\cos\nu - e^2 - e\cos\nu)(1 - e^2)^{1/2} = (1 + e\cos\nu)^2$$

$$\cos\nu = \frac{1}{e}\left((1 - e^2)^{3/4} - 1\right)$$

We are done!

By substituting the eccentricity of Earth's orbit, the anomaly values $90.7°$ and $277.3°$ are obtained, which correspond respectively to April 15 and December 27. The figure below shows these points.

We can infer that the critical points of the equation-of-time graph are mainly produced by the axial tilt, since the effect of orbital eccentricity. In reality, Earth's orbital eccentricity does not have a large effect on the equation of time.

Now we can examine the special case, that is, when the eccentricity is zero and only the axial tilt is present. We again start from the equation of time relation:

$$Eqt = (M - \nu_0) - \tan^{-1}(\cos\varepsilon \tan(\nu - \nu_0))$$
$$\tan(M - \nu_0 - Eqt) = \cos\varepsilon \tan(\nu - \nu_0)$$

We take the derivative:

$$(1 + \tan^2(M - \nu_0 - Eqt))\left(1 - \frac{dEqt}{dM}\right) = \cos\varepsilon(1 + \tan^2(\nu - \nu_0))\frac{d\nu}{dM}$$

Setting $\frac{dEqt}{dM} = 0$ and substituting the above relation for $\tan(M - \nu_0 - Eqt)$:

$$\frac{\cos\varepsilon\left(1 + \tan^2(\nu - \nu_0)\right)}{1 + \cos^2\varepsilon\tan^2(\nu - \nu_0)} = \frac{dM}{d\nu}$$

We substitute both sides of the equation using the relations from the previous section:

$$\frac{\cos\varepsilon\left(1 + \tan^2(\nu - \nu_0)\right)}{1 + \cos^2\varepsilon\tan^2(\nu - \nu_0)} = (1 - e^2)^{3/2}\frac{1 + e}{1 + e\cos\nu}\left(\frac{1 + e\cos\nu}{1 + e\cos\nu}\right)\beta$$

$$\frac{\cos\varepsilon}{\cos^2(\nu - \nu_0)} = \frac{1 - e^2}{1 + e\cos\nu}\left(\frac{1 - e^2}{1 + e\cos\nu}\right)^{1/2}\left(1 + \cos^2\varepsilon\tan^2(\nu - \nu_0)\right)$$

With some algebraic manipulation, we arrive at the following relation:

$$\cos\varepsilon(1 + e\cos\nu)^2 = (1 - e^2)^{3/2}\left(\cos^2(\nu - \nu_0) + \cos^2\varepsilon\sin^2(\nu - \nu_0)\right)$$

We can also rewrite this relation in terms of the Sun's ecliptic longitude:

$$\lambda_\odot = \nu - \nu_0$$

$$\cos\varepsilon\left(1 + e\cos(\lambda_\odot + \nu_0)\right)^2 = (1 - e^2)^{3/2}\left(\cos^2\lambda_\odot + \cos^2\varepsilon\sin^2\lambda_\odot\right)$$

This relation is not quadratic, but we can solve it using numerical methods such as Newton's method. The solutions correspond to the same critical points of the equation-of-time graph that were obtained previously. Solving the equation numerically for the Sun's ecliptic longitude, we obtain approximately

$$\lambda_\odot \approx 29.8°,\ 123.3°,\ 211.0°,\ 332.7°,$$

which are the same values extracted from the equation-of-time graph.

Determination of the Observer's Latitude from the Analemma

The analemma provides a geometric method for determining the observer's latitude by comparing the solar altitude at two specific points on the curve. In particular, we consider the two extrema of the analemma corresponding to the solstices, when the Sun reaches its maximum and minimum declination.
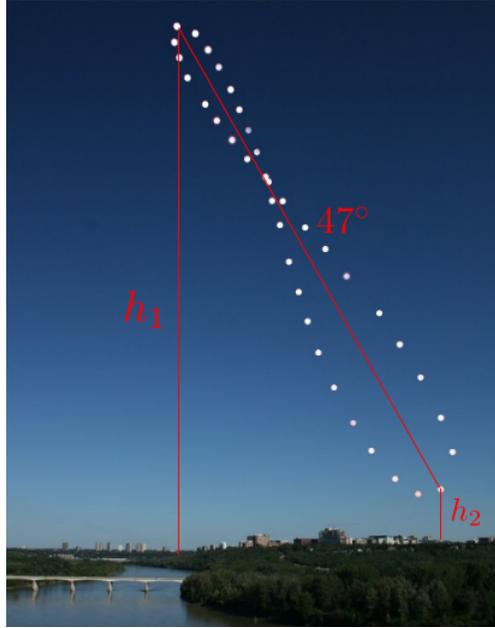
Let the observer's latitude be denoted by φ. Consider the Sun observed at the same local clock time on two different days, corresponding to declinations δ_1 and δ_2, with measured altitudes h_1 and h_2, respectively. The transformation between the equatorial and horizontal coordinate systems gives the standard relation

$$\sin h = \sin \delta \sin \varphi + \cos \delta \cos \varphi \cos \tau,$$

where τ is the hour angle of the Sun at the time of observation.
Applying this relation to the two observations, we obtain

$$\sin h_1 = \sin \delta_1 \sin \varphi + \cos \delta_1 \cos \varphi \cos \tau,$$
$$\sin h_2 = \sin \delta_2 \sin \varphi + \cos \delta_2 \cos \varphi \cos \tau.$$

Rearranging each equation and dividing by the corresponding $\cos \delta$, we find

$$\frac{\sin h_1 - \sin \delta_1 \sin \varphi}{\cos \delta_1} = \frac{\sin h_2 - \sin \delta_2 \sin \varphi}{\cos \delta_2}.$$

Cross-multiplying yields

$$\cos \delta_2 \sin h_1 - \cos \delta_2 \sin \delta_1 \sin \varphi = \cos \delta_1 \sin h_2 - \cos \delta_1 \sin \delta_2 \sin \varphi.$$

We now specialize to the case of the solstices. At these points, the Sun's declination satisfies

$$\delta_1 = -\varepsilon,$$
$$\delta_2 = +\varepsilon,$$

where ε is the obliquity of the ecliptic, i.e., the angle between the Earth's equatorial plane and the ecliptic. Substituting these values and using $\cos(-\varepsilon) = \cos\varepsilon$, the previous equation becomes

$$\cos\varepsilon\sin h_1 + \cos\varepsilon\sin\varepsilon\sin\varphi = \cos\varepsilon\sin h_2 - \cos\varepsilon\sin\varepsilon\sin\varphi.$$

Dividing through by $\cos\varepsilon$ and rearranging terms, we obtain

$$\sin h_2 - \sin h_1 = 2\sin\varepsilon\sin\varphi.$$

Finally, solving for the observer's latitude, we arrive at

$$\boxed{\sin\varphi = \frac{\sin h_2 - \sin h_1}{2\sin\varepsilon}.}$$

This expression shows that the latitude can be determined directly from the difference in the observed solar altitudes at the two solstitial points of the analemma. Notably, the dependence on the hour angle τ cancels out, provided the observations are made at the same local time on both days.

Angle Between the Tangent Lines of the Analemma

As a final topic, we wish to determine the angle between the tangent lines to the analemma at two points. To first order, the angle between the tangent lines can be obtained from the slope of the line connecting two successive points. The smaller the distance between these two points, the more accurate this approximation becomes. Here, we assume the distance between two points to be one day. In the figure below, the actual position of the Sun on the analemma over two successive days is shown.
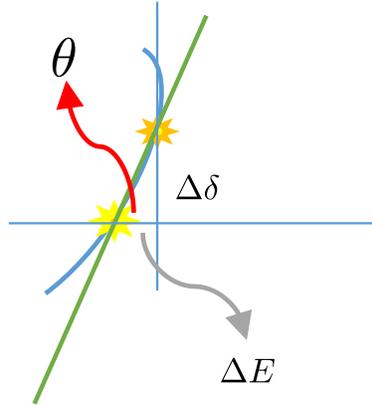
From here, the angle of the slope of the line connecting the two points is given by

$$\theta = \tan^{-1}\left(\frac{\Delta\delta}{\Delta E}\right).$$

Our goal is to find this angle at the given point. The solar declination at this point is approximately known, so by performing the calculations using the data below, we obtain:

	Turning Points of the Analemma			
	First Turning Point		**Second Turning Point**	
Date	April 22	April 23	September 6	September 7
Equation of Time (min)	+0.88	+0.75	−1.09	−0.78
Declination (deg)	8.65°	9.01°	9.39°	9.03°

Table 1: Equation of time (minutes) and solar declination (degrees) for two consecutive days near the two turning points of the analemma.

Using these values, we obtain

$$\theta_1 = \tan^{-1}\left(\frac{\Delta\delta}{\Delta E}\right) = \tan^{-1}\left(\frac{9.01 - 8.65}{\frac{-0.75 + 0.88}{4}}\right) = 84.84°,$$

$$\theta_2 = \tan^{-1}\left(\frac{9.03 - 9.39}{\frac{-0.78 + 1.09}{4}}\right) = 102.15°.$$

Therefore, the angle between the two branches of the analemma is

$$\Delta\theta = \theta_2 - \theta_1 = 102.15° - 84.84° = 17.31°.$$

If you remember, when we ignored the eccentricity, this angle was exactly ε. But now it is less than that. Of course,you could have guessed this from the graphs we drew.

Finally, we will give some examples to get more familiar with how to use time modulation.

Example 5.2

In a city with geographical coordinates

$$\varphi = 35°30', \qquad L = 58°40',$$

On 16 Shahrivar, at what official clock time does the star *Shaula* rise? Ignore variations of the Equation of Time over the course of the day, and assume the city's elevation above sea level to be 1000 meters. The equatorial coordinates of the star Shaula are:

$$\alpha = 6^{\mathrm{h}}45^{\mathrm{m}}, \qquad \delta = -16°44'.$$

We can easily obtain the hour angle of Shaula at the moment of rising, and then, by determining the local sidereal time, obtain the result.

$$\mathrm{LST} = HA_s + RA_s$$

We also write the sidereal time relation for the mean Sun:

$$\mathrm{LST} = HA_{\mathrm{MS}} + RA_{\mathrm{MS}}$$

To find the local time, we need the hour angle of the mean Sun; therefore, from the two relations above, we have:

$$HA_{\mathrm{MS}} = HA_s + (RA_s - RA_{\mathrm{MS}})$$

The right ascension of the mean Sun can also be calculated easily, and there is no need to use the Equation of Time here.

Now we proceed quickly to the calculations. To find the hour angle of Shaula at rising, we use the well-known spherical triangle PZS.
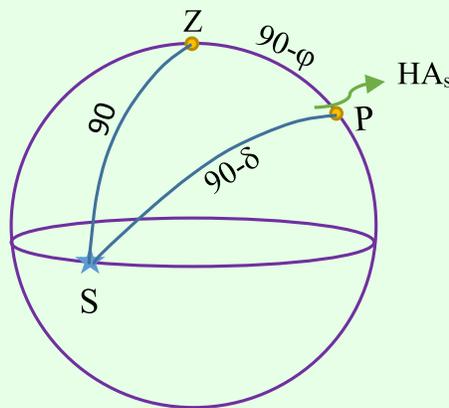
We assumed the side ZS to be 90°. You may ask: what happens after considering the city's elevation? Due to the city's elevation, the apparent horizon is lower, and therefore the side

ZS should be slightly larger than $90°$. Your question is correct. However, elevation only has an effect if we are on a mountain or at high altitude, such that our elevation is higher than the surrounding area, not relative to sea level. In this problem, we do not have accurate information about the surroundings, so we assume that the entire region is at the same elevation.

$$\cos 90 = \cos(90 - \varphi)\cos(90 - \delta) + \sin(90 - \varphi)\sin(90 - \delta)\cos HA_s$$
$$\cos HA_s = -\tan\varphi\tan\delta = -\tan(-16°45')\tan(35°30')$$
$$HA_s = -77°36' = -5^h10^m$$

Note that the hour angle of sunrise is always negative; we have included the negative sign ourselves!

Now we compute the hour angle of the mean Sun. We perform this calculation for 16 Shahrivar. The mean anomaly is:

$$M = \frac{360°}{365.25} \times (16 + 5 \times 31 + 75.5) = 242°57'$$

Recall from earlier that the true anomaly of the vernal equinox is $\nu_0 = 76°20'$. Thus:

$$RA_{\mathrm{MS}} = 242°57' - 76°20' = 166°37' = 11^h6^m$$

By substituting into the relation we derived, the hour angle of the mean Sun is obtained as:

$$HA_{\mathrm{MS}} = -5^h10^m + (6^h45^m - 11^h6^m) = -9^h31^m$$

Since this time is after midnight, we now have the local time:

$$\mathrm{LMT} = 12^h + HA_{\mathrm{MS}} = 2^h29^m$$

The calculation is not finished yet. We must convert this time to official (civil) time, meaning standard noon. For Iran, the standard meridian longitude is $L_{\mathrm{std}} = 52°30'$. Therefore, the time difference relative to standard noon is:

$$\Delta t = \frac{L_{\mathrm{std}} - L}{15} = \frac{52°30' - 58°40'}{15} = -0^h24^m$$

We add this value to the local time:

$$\mathrm{MT} = 2^h29^m - 0^h24^m = 2^h5^m$$

But the work is still not finished! We know that in Iran, during the first six months of the year, daylight saving time is applied. Therefore, the final time obtained is 3^h5^m. Astronomy software reports the rising time of Shaula as 3^h7^m on 15 Shahrivar, and 3^h3^m on 16 Shahrivar. It seems that we have performed the calculation correctly!

Example 5.3

For the city considered in the previous example, determine the time of solar noon on the first day of Azar, and then compute the remaining religious (prayer) times.

Solar noon occurs when the Sun reaches its maximum altitude, which approximately coincides with its meridian transit. At this moment,

$$\mathrm{HA}_\odot = 0.$$

Using the relation between true and mean solar time,

$$\mathrm{RA}_\odot + \mathrm{HA}_\odot = \mathrm{RA}_{\mathrm{MS}} + \mathrm{HA}_{\mathrm{MS}},$$

we obtain

$$\mathrm{HA}_{\mathrm{MS}} = \mathrm{RA}_\odot - \mathrm{RA}_{\mathrm{MS}} = -Eqt.$$

Thus,

$$\mathrm{LMT} = 12 + \mathrm{HA}_{\mathrm{MS}} = 12 - Eqt.$$

The mean anomaly for the given date is

$$M = \frac{2\pi}{365.25}(186 + 2 \times 30 + 75.5) = 5.5306^{\mathrm{r}} = 316°53'.$$

We solve Kepler's equation iteratively:

$$E_{i+1} = E_i - \frac{E_i - M - e \sin E_i}{1 - e \cos E_i}.$$

Starting from $E_1 = M$, convergence is achieved at

$$E = 5.5190.$$

The true anomaly is

$$\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \quad \nu = 315°33'.$$

The Sun's ecliptic longitude is

$$\lambda_\odot = \nu - \nu_0 = 239°13'.$$

The Sun's right ascension is

$$\mathrm{RA}_\odot = \tan^{-1}(\cos\varepsilon\tan\lambda_\odot) = 237°0'.$$

The mean Sun's right ascension is

$$\mathrm{RA}_{\mathrm{MS}} = M - \nu_0 = 240°33'.$$

Hence,

$$Eqt = \mathrm{RA}_{\mathrm{MS}} - \mathrm{RA}_\odot = 3°33' = 14^{\mathrm{m}}.$$

Solar noon occurs at

$$\mathrm{MT} = 11^{\mathrm{h}}46^{\mathrm{m}} - 0^{\mathrm{h}}24^{\mathrm{m}} = 11^{\mathrm{h}}22^{\mathrm{m}}.$$

Having obtained the time of solar noon, we can compute the remaining religious times; however, first we must calculate the Sun's declination:

$$\frac{\sin \delta_\odot}{\sin \varepsilon} = \frac{\sin \lambda_\odot}{\sin 90} \implies \delta_\odot = \sin^{-1}(\sin \varepsilon \sin \lambda_\odot) = -19°59'$$

In this part, we work with the spherical triangle PZS and therefore write the cosine rule as:

$$\cos(90 - a) = \cos(90 - \varphi)\cos(90 - \delta) + \sin(90 - \varphi)\sin(90 - \delta)\cos HA_s$$

Solving this equation for the hour angle, we obtain:

$$\cos HA_s = \frac{\sin a - \sin \varphi \sin \delta}{\cos \varphi \cos \delta}$$

The religious times are determined in the following manner:
Morning prayer (Fajr): the Sun is at an altitude of $18°$ below the horizon.
Sunrise and sunset: the Sun is on the horizon.
Evening prayer (Maghrib): the Sun is at an altitude of $4.5°$ below the horizon.
Accordingly, the Sun's hour angle at each of the four times is obtained as:

$$HA_{\text{Fajr}} = -6^h33^m, \quad HA_{\text{Sunrise}} = -5^h00^m, \quad HA_{\text{Sunset}} = 5^h00^m, \quad HA_{\text{Maghrib}} = 5^h24^m$$

By adding each of these hour angles to the time of solar noon, the time of each event is obtained:

Fajr	Sunrise	Solar Noon	Sunset	Maghrib
4^h49^m	6^h22^m	11^h22^m	16^h22^m	16^h46^m

Problems

1. An observer at latitude $25.5°$ north takes a photograph of the Sun every day at a fixed local time and then superimposes the images. As expected, an analemma is formed. It is observed that all points of the analemma lie above the horizon, and only one point of the analemma is tangent to the horizon.

   (a) What is the angle between the axis of the analemma and the horizon?
   (b) What is the angle between the lowest point of the analemma and the horizon?

2. We expect that the Sun's declination decreases after the winter solstice. Therefore, for a given observer, the Sun should rise later each day after the first day of winter. However, this is not what actually happens.

   (a) Explain why this expectation is incorrect, using your own calculations.
   (b) Assuming a circular orbit for the Earth around the Sun, determine the number of days per year on which the Sun rises later than the previous day.
   (c) Does the same phenomenon occur in summer?

3. According to the saying "the first hour after sunset," at what time does the shadow of a gnomon return to its shortest length in the afternoon? Determine this *true solar time* for the previous example.

4. In the city of Sujawad (Romania), with geographical coordinates

$$\varphi = 47°38', \qquad L = 26°15',$$

an observer watches the meridian transit of the star *Shaula*. The clock shows Romanian civil time with

$$\text{time zone} = +2\,\text{h},$$

and indicates 00:00. Determine the date of this event. (Note: Romania's National Olympiad consists of two rounds and a final.)

5. An observer in the Southern Hemisphere sees the rising of the south celestial pole and wonders how this is possible. If, instead of rotating around the south celestial pole, the observer moves along the surface of the Earth while always facing south with constant angular speed, describe and sketch the apparent motion of the south celestial pole in the sky. Plot its path over a full day and determine its direction and speed when it crosses the horizon for the first time. (This was a question from the World Astronomy Olympiad.)

Computational Exercises

1. Write a program that computes the Equation of Time for every day of the year. Then attempt to reproduce the plots presented in this text. Compare your results with the figures shown here, which were generated using reliable software based on observational data.

2. Obtain a table of solar times (for example, using Excel or similar software). Plot the time of sunrise as a function of the day of the year. Then, by interpreting the resulting plot, explain why daylight saving time (summer time) is used during the first half of the year.

3. In this text, the Equation of Time was plotted as a function of the day of the year in Cartesian coordinates. Now plot the same curve in polar coordinates. Also try plotting the negative values of the Equation of Time—this may lead to new insights.